Recovery of algebraic-exponential data from moments

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★ Part of this work is joint with M. Putinar
Motivation

- An important property of Positively Homogeneous Functions (PHF)
- Some properties (convexity, polarity)
- Sub-level sets of minimum volume containing $K$
- Exact reconstruction from moments
- Recovery of the defining function of a semi-algebraic set
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Reconstruction of a shape $K \subset \mathbb{R}^n$ (convex or not) from knowledge of finitely many moments

$$y_\alpha = \int_{K} x_1^{\alpha_1} \cdots x_n^{\alpha_n} \, dx, \quad \alpha \in \mathbb{N}^n_d,$$

for some integer $d$, is a difficult and challenging problem!

EXACT recovery of $K$

from $y = (y_\alpha), \alpha \in \mathbb{N}^n_d$, is even more difficult and challenging!
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### Examples of exact recovery:

- **Quadrature (planar) Domains** in \((\mathbb{R}^2)\) (Gustafsson, He, Milanfar and Putinar (Inverse Problems, 2000))
  - via an exponential transform

- **Convex Polytopes** in \(\mathbb{R}^n\) (Gravin, L., Pasechnik and Robins (Discrete & Comput. Geometry (2012))
  - Use Brion-Barvinok-Khovanski-Lawrence-Pukhlikov moment formula for projections \[ \int_P \langle c, x \rangle^j \, dx \] combined with a **Prony**-type method to recover the vertices of \(P\).
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Examples of exact recovery:

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Approximate recovery can be done in multi-dimensions (Cuyp, Golub, Milanfar and Verdonk, 2005) via:

- (multi-dimensional versions of) homogeneous Padé approximants applied to the Stieltjes transform.
- Cubature formula at each point of grid
- Solving a linear system of equations to retrieve the indicator function of $K$
This talk: I

- Exact recovery.
- \( K = \{ x \in \mathbb{R}^n : g(x) \leq 1 \} \) has finite Lebesgue volume.
- \( g \) is a nonnegative homogeneous polynomial.
- Data are finitely many moments:

\[
y_\alpha = \int_K x^\alpha \, dx, \quad \alpha \in \mathbb{N}_d^n.
\]

- Also works for Quasi-homogeneous polynomials, i.e., when

\[
g(\lambda^{u_1} x_1, \ldots, \lambda^{u_n} x_n) = \lambda \, g(x), \quad x \in \mathbb{R}^n, \quad \lambda > 0
\]

for some vector \( u \in \mathbb{Q}^n \).

\((d\text{-Homogeneous } = u\text{-quasi homogeneous with } u_i = \frac{1}{d} \text{ for all } i)\).
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Positively Homogeneous functions (PHF) form a wide class of functions encountered in many applications. As a consequence of homogeneity, they enjoy very particular properties, and among them the celebrated and very useful Euler’s identity which allows to deduce additional properties of PHFs in various contexts.

Another (apparently not well-known) property of PHFs yields surprising and unexpected results, some of them already known in particular cases.

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A little detour

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The case of homogeneous polynomials is even more interesting!
So we are now concerned with PHFs, their sublevel sets and in particular, the integral

\[ y \mapsto I_{g,h}(y) := \int_{\{x : g(x) \leq y\}} h(x) \, dx, \]

as a function \( I_{g,h} : \mathbb{R}_+ \to \mathbb{R} \) when \( g, h \) are PHFs.

With \( y \) fixed, we are also interested in

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now as a function of \( g \), especially when \( g \) is a nonnegative homogeneous polynomial.

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Nonnegative homogeneous polynomials are particularly interesting as they can be used to approximate norms; see e.g. Barvinok.
Interestingly, the latter integral is related in a simple and remarkable manner to the non-Gaussian integral
\[ \int_{\mathbb{R}^n} h \exp(-g) \, dx. \]

Functional integrals appear frequently in quantum Physics where a challenging issue is to provide exact formulas for \( \int \exp(-g) \, dx \), the most well-known being when \( \text{deg} \, g = 2 \),
\[ d = 2 \Rightarrow \int \exp(-g) \, dx = \frac{\text{Cte}}{\sqrt{\det(g)}}. \]

Observe that \( \det(g) \) is an algebraic invariant of \( g \),
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The key tools are discriminants and $SL(n)$-invariants.

An integral

$$J(g) := \int \exp(-g) \, dx$$

is called a discriminant integral.

Write the polynomial $g \in \mathbb{R}[x]$ in the monomial basis as

$$g(x) = \sum_{a \in \mathbb{N}^n} g_a x^a$$

$$= \sum_{a \in \mathbb{N}^n} g_{a_1 \cdots a_n} x_1^{a_1} \cdots x_n^{a_n}$$

for finitely many coefficients $(g_a)$. 
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for finitely many coefficients ($g_a$).
Integral discriminants satisfy WARD Identities

\[
\left( \frac{\partial}{\partial g_{a_1 \ldots a_n}} \frac{\partial}{\partial g_{b_1 \ldots b_n}} - \frac{\partial}{\partial g_{c_1 \ldots c_n}} \frac{\partial}{\partial g_{d_1 \ldots d_n}} \right) \cdot J(g) = 0,
\]

where \( a_i + b_i = c_i + d_i \) for all \( i \).

which permits to obtain exact formulas in low-dimensional cases in terms of algebraic invariants of \( g \). See e.g. Morosov and Shakirov\(^1\)

\(^1\)New and old results in Resultant theory, arXiv.0911.5278v1.
In particular, as a by-product in the important particular case when \( h = 1 \), they have proved that for all forms \( g \) of degree \( d \),

\[
\text{Vol} \left( \{ x : g(x) \leq 1 \} \right) = \int_{\{ x : g(x) \leq 1 \}} dx = \text{cte}(d) \cdot \int_{\mathbb{R}^n} \exp(-g) dx,
\]

where the constant depends only on \( d \) and \( n \).

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In fact, a formula of exactly the same flavor was already known for convex sets, and was the initial motivation of our work. Namely, if $C \subset \mathbb{R}^n$ is convex, its support function

$$x \mapsto \sigma_C(x) := \sup \{ x^T y : y \in C \},$$

is a PHF of degree 1, and the polar $C^\circ \subset \mathbb{R}^n$ of $C$ is the convex set $\{ x : \sigma_C(x) \leq 1 \}$.

Then...

$$\text{vol} \left( C^\circ \right) = \frac{1}{n!} \int_{\mathbb{R}^n} \exp(-\sigma_C(x)) \, dx, \quad \forall C.$$
I. An important property of PHF’s

We are interested in integrals of the form:

\[ \int_{\mathbb{R}^n} \phi(g(x)) \, dx \quad \text{with} \quad \phi : \mathbb{R}_+ \to \mathbb{R}. \]

For instance

- \( t \mapsto \phi(t) := l_{[0,1]}(t) \) yields \( \int_{\{x:0 \leq g(x) \leq 1\}} dx \)
- \( t \mapsto \phi(t) := t \, l_{[0,1]}(t) \) yields \( \int_{\{x:0 \leq g(x) \leq 1\}} g(x) \, dx \)
- \( t \mapsto \phi(t) := \exp(-t) \) yields \( \int_{\mathbb{R}^n} \exp(-g(x)) \, dx \)
- \( t \mapsto \phi(t) := t \, \exp(-t) \) yields \( \int_{\mathbb{R}^n} g(x) \, \exp(-g(x)) \, dx \)

etc.
Theorem

Let $\phi : \mathbb{R}_+ \to \mathbb{R}$ be a measurable mapping, and let $g \geq 0$ and $h$ be PHFs of respective degree $0 \neq d, p \in \mathbb{Z}$ and such that

$$\int |h| \exp(-g) \, dx$$

is finite,

$$\int_{\mathbb{R}^n} \phi(g(x)) \, h(x) \, dx = C(\phi, d, p) \cdot \int_{\mathbb{R}^n} h \exp(-g) \, dx,$$

where the constant $C(\phi, d, p)$ depends only on $\phi, d, p$.

In particular, if the sublevel set $\{ x : g(x) \leq 1 \}$ is bounded, then

$$\int_{\{ x : g(x) \leq y \}} h \, dx = \frac{y^{(n+p)/d}}{\Gamma(1 + (n + p)/d)} \int_{\mathbb{R}^n} h \exp(-g) \, dx,$$

with $\Gamma$ being the standard Gamma function.
Proof for nonnegative $h$

For simplicity assume that $g(x) > 0$ if $x \neq 0$. With $z = (z_1, \ldots, z_{n-1})$, do the change of variable $x_1 = t$, $x_2 = t z_1$, $x_n = t z_{n-1}$ so that one may decompose $\int_{\mathbb{R}^n} \phi(g(x)) \ h(x) dx$ into the sum

$$
\int_{\mathbb{R}_+ \times \mathbb{R}^{n-1}} t^{n+p-1} \phi(t^d g(1, z)) \ h(1, z) \ dt \ dz \\
+ \int_{\mathbb{R}_+ \times \mathbb{R}^{n-1}} t^{n+p-1} \phi(t^d g(-1, -z)) \ h(-1, z) \ dt \ dz,
$$

$$
= \int_{\mathbb{R}^{n-1}} \left( \int_0^\infty t^{n+p-1} \phi(t^d g(1, z)) \ dt \right) \ h(1, z) \ dz \\
+ \int_{\mathbb{R}^{n-1}} \left( \int_0^\infty t^{n+p-1} \phi(t^d g(-1, -z)) \ dt \right) \ h(-1, -z) \ dz,
$$

where the last two integrals are obtained from the sum of the previous two by using Tonelli’s Theorem.
Next, with the change of variable \( u = t g(1, z)^{1/d} \) and 
\( u = t g(-1, -z)^{1/d} \)

\[
\int_{\mathbb{R}^n} \phi(g(x)) h(x) \, dx = \left( \int_{\mathbb{R}_+} u^{n+p-1} \phi(u^d) \, du \right) \cdot A(g, h),
\]

with

\[
A(g, h) = \int_{\mathbb{R}^{n-1}} \left( \frac{h(1, z)}{g(1, z)^{(n+p)/d}} + \frac{h(-1, -z)}{g(-1, -z)^{(n+p)/d}} \right) \, dz.
\]
Choosing $\phi(t) = \exp(-t)$ on $[0, +\infty)$ yields:
\[
\int_{\mathbb{R}^n} \exp(-g(x)) h(x) \, dx = \frac{\Gamma(1 + (n+p)/d)}{n + p} \cdot A(g, h),
\]
whereas, choosing $\phi(t) = I_{[0,1]}(t)$ on $[0, +\infty)$ yields:
\[
\int_{\{x : g(x) \leq 1\}} h(x) \, dx = \frac{1}{n + p} \cdot A(g, h),
\]
And so in particular, whenever \( g \) is nonnegative and \( \{ x : g(x) \leq 1 \} \) has finite Lebesgue volume:

**Theorem**

If \( g, h \) are PHFs of degree \( 0 < d \) and \( p \) respectively, then:

\[
\int_{\{x : g(x) \leq y\}} h \, dx = \frac{y^{(n+p)/d}}{\Gamma(1 + (n + p)/d)} \int_{\mathbb{R}^n} \exp(-g) \, h \, dx
\]

\[
\text{vol} \left( \{ x : g(x) \leq y \} \right) = \frac{y^{n/d}}{\Gamma(1 + n/d)} \int_{\mathbb{R}^n} \exp(-g) \, dx
\]
An alternative proof

Let \( g, h \) be nonnegative so that \( I_{g,h}(y) \) vanishes on \(( -\infty, 0 ] \). For \( 0 < \lambda \in \mathbb{R} \), its Laplace transform

\[
\lambda \mapsto \mathcal{L}_{I_{g,h}}(\lambda) = \int_0^\infty \exp(-\lambda y) I_{g,h}(y) \, dy
\]

reads:

\[
\mathcal{L}_{I_{g,h}}(\lambda) = \int_0^\infty \exp(-\lambda y) \left( \int_{\{x: g(x) \leq y\}} h \, dx \right) \, dy
\]

\[
= \int_{\mathbb{R}^n} h(x) \left( \int_0^\infty \exp(-\lambda y) \, dy \right) \, dx \quad [\text{by Fubini}]
\]

\[
= \frac{1}{\lambda} \int_{\mathbb{R}^n} h(x) \exp(-\lambda g(x)) \, dx
\]

\[
= \frac{1}{\lambda^{1+(n+p)/d}} \int_{\mathbb{R}^n} h(z) \exp(-g(z)) \, dz \quad [\text{by homog}]
\]

\[
= \int_{\mathbb{R}^n} h(z) \exp(-g(z)) \, dz
\]

\[
= \frac{1}{\Gamma(1 + (n + p)/d)} \, \mathcal{L}_{y^{(n+p)/d}}(\lambda).
\]
And so, by analyticity and the Identity theorem of analytical functions

\[ I_{g,h}(y) = \frac{y^{(n+p)/d}}{\Gamma \left( 1 + \frac{n + p}{d} \right)} \int_{\mathbb{R}^n} h(x) \exp(-g(x)) \, dx, \]
II. Approximating a non gaussian integral

Hence computing the non Gaussian integral $\int \exp(-g) \, dx$

reduces to computing the volume of the level set

$G := \{ x : g(x) \leq 1 \}$,

... which is the same as solving the optimization problem:

$$\max_{\mu} \quad \mu(G)$$

s.t. \hspace{1cm} \mu + \nu = \lambda \hspace{1cm} \mu(B \setminus G) = 0$

where:

- $B$ is a box $[-a, a]^n$ containing $G$ and
- $\lambda$ is the Lebesgue measure.
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- $B$ is a box $[-a, a]^n$ containing $G$ and
- $\lambda$ is the Lebesgue measure.
and we know how to approximate as closely as desired $\mu(G)$ and any FIXED number of moments of $\mu$, by solving an appropriate hierarchy of semidefinite programs (SDP).

(see: *Approximate volume and integration for basic semi algebraic sets*, Henrion, Lasserre and Savorgnan, SIAM Review 51, 2009.)

However ...

the resulting SDPs are numerically difficult to solve.

Solving the dual reduces to approximating the indicator function $I(G)$ by polynomials of increasing degrees $\rightarrow$ Gibbs effect, etc.
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However ... the resulting SDPs are numerically difficult to solve.

Solving the dual reduces to approximating the indicator function $I(G)$ by polynomials of increasing degrees → Gibbs effect, etc.
Let $G \subseteq B := [-1, 1]^n$ (possibly after scaling), and let $z = (z_\alpha)$, $\alpha \in \mathbb{N}_2^n$, be the moments of the Lebesgue measure $\lambda$ on $B$.

Solve the hierarchy of semidefinite programs:

$$\rho_k = \max_{y_0} \quad \text{s.t.} \quad M_k(y), M_k(v) \succeq 0,$$

$$M_{k-\lceil(d)/2\rceil}(gy) \succeq 0,$$

$$M_{k-1}((1 - x_i^2)v) \succeq 0, \quad i = 1, \ldots, n,$$

$$y_\alpha + v_\alpha = z_\alpha, \quad \alpha \in \mathbb{N}_2^n$$

for some moment and localizing matrices $M_k(y)$ and $M_k(g, y)$. The linear constraints $y_\alpha + v_\alpha = z_\alpha$ for all $\alpha \in \mathbb{N}_2^n$ "ensure" $\mu + \nu = \lambda$, while the "$\succeq 0$" constraints "ensure" $\text{supp } \mu = G$ and $\text{supp } \nu = B$. 
Let $G \subseteq B := [-1, 1]^n$ (possibly after scaling), and let $z = (z_\alpha)$, $\alpha \in \mathbb{N}_{2k}^n$, be the moments of the Lebesgue measure $\lambda$ on $B$.

Solve the hierarchy of semidefinite programs:

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for some **moment** and **localizing** matrices $M_k(y)$ and $M_k(g, y)$.

- The linear constraints $y_\alpha + v_\alpha = z_\alpha$ for all $\alpha \in \mathbb{N}_{2k}^n$ “ensure” $\mu + \nu = \lambda$, while the “$\succeq 0$” constraints “ensure” $\text{supp } \mu = G$ and $\text{supp } \nu = B$. 

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Corollary

If $g$ has degree $d$ and $G$ has finite volume then

\[
\underbrace{\int_{\{x : g(x) \leq y\}} \exp(-g) \, dx}_{\int_{\mathbb{R}^n} \exp(-g) \, dx} = \underbrace{\int_{0}^{y} t^{n/d-1} \exp(-t) \, dt}_{\int_{0}^{\infty} t^{n/d-1} \exp(-t) \, dt} = \underbrace{\int_{0}^{y} t^{n/d-1} \exp(-t) \, dt}_{\Gamma(n/d)}
\]

expresses how fast $\mu(\{x : g(x) \leq y\})$ goes to $\mu(\mathbb{R}^n)$ as $y \to \infty$, for the Borel measure $d\mu = \exp(-g) \, dx$.

It is like for the Gamma function $\Gamma(n/d)$ when approximated by $\int_{0}^{y} t^{n/d-1} \exp(-t) \, dt$. 
III. Convexity

An interesting issue is to analyze how the Lebesgue volume
\( \text{vol} \{ x \in \mathbb{R}^n : g(x) \leq 1 \} \), (i.e. \( \text{vol} (G) \)) changes with \( g \).

Corollary

Let \( h \) be a PHF of degree \( p \) and let \( C_d \subset \mathbb{R}[x]_d \) be the convex cone of homogeneous polynomials of degree at most \( d \) such that \( G \) is bounded. Then the function \( f_h : C_d \to \mathbb{R}, \)

\[
  g \mapsto f_h(g) := \int_G h \, dx, \quad g \in C_d,
\]

- is a PHF of degree \(- (n + p)/d\),
- **convex** whenever \( h \) is nonnegative and strictly convex if \( h > 0 \) on \( \mathbb{R}^n \setminus \{0\} \).
Moreover, if \( h \) is continuous and \( \int |h| \exp(-g) \, dx < \infty \) then:

\[
\frac{\partial f_h(g)}{\partial g_\alpha} = \frac{-1}{\Gamma(1 + (n + p)/d)} \int_{\mathbb{R}^n} x^\alpha h \exp(-g) \, dx
\]

\[
= \frac{-\Gamma(2 + (n + p)/d)}{\Gamma(1 + (n + p)/d)} \int_G x^\alpha h \, dx
\]

\[
\frac{\partial^2 f_h(g)}{\partial g_\alpha \partial g_\beta} = \frac{-1}{\Gamma(1 + (n + p)/d)} \int_{\mathbb{R}^n} x^{\alpha + \beta} h \exp(-g) \, dx
\]
**PROOF:** Just use

\[
\int_{\{x : g(x) \leq 1\}} h \, dx = \frac{1}{\Gamma(1 + (n + p)/d)} \int_{\mathbb{R}^n} h \exp(-g) \, dx
\]

Notice that proving convexity **directly** would be non trivial but becomes easy when using the previous lemma!
PROOF: Just use

\[ \int_{\{x : g(x) \leq 1\}} h \, dx = \frac{1}{\Gamma(1 + (n + p)/d)} \int_{\mathbb{R}^n} h \exp(-g) \, dx \]

Notice that proving convexity directly would be non trivial but becomes easy when using the previous lemma!
For a set $C \subset \mathbb{R}^n$, recall:

- The support function $x \mapsto \sigma_C(x) := \sup \{ x^T y : y \in C \}$
- The POLAR $C^\circ := \{ x \in \mathbb{R}^n : \sigma_C(x) \leq 1 \}$
- and for a PHF $g$ of degree $d$, its Legendre-Fenchel conjugate $g^*(x) = \sup \{ x^T y - g(y) \}$ is a PHF of degree $q$ with $\frac{1}{d} + \frac{1}{q} = 1$. 
Lemma

Let $g$ be a closed proper convex PHF of degree $1 < d$ and let $G = \{ x : g(x) \leq 1/d \}$. Then:

$$G^\circ = \{ x \in \mathbb{R}^n : g^*(x) \leq 1/q \}$$

$$\text{vol}(G) = \frac{p^{-n/p}}{\Gamma(1 + n/p)} \int \exp(-g) \, dx$$

$$\text{vol}(G^\circ) = \frac{q^{-n/q}}{\Gamma(1 + n/q)} \int \exp(-g^*) \, dx$$

→ yields completely symmetric formulas for $g$ and its conjugate $g^*$. 

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Examples

- $g(x) = |x|^3$ so that $g^*(x) = \frac{2}{3\sqrt{3}}|x|^{3/2}$. And so
  $$G = [-3^{-1/3}, 3^{-1/3}]; \quad G^\circ = [-3^{1/3}, 3^{1/3}].$$

- TV screen: $g(x) = x_1^4 + x_2^4$ so that $g^*(x) = 4^{-4/3}3(x_1^{4/3} + x_2^{4/3})$. And,
  $$G = \{x : x_1^2 + x_2^4 \leq \frac{1}{4}\}; \quad G^\circ = \{x : x_1^{4/3} + x_2^{4/3} \leq 4^{1/3}\}.$$

- $g(x) = |x|$ so that $d \not> 1$, and $g^*(x) = 0$ if $x \in [-1, 1]$, and $+\infty$ otherwise. Hence $G = \{x : |x| \leq 1\} = [-1, 1]$ and with $q = +\infty$,
  $$G^\circ = [-1, 1] = \{x : g^*(x) \leq \frac{1}{q} = 0\}.$$
IV. A variational property of homogeneous polynomials

Let $\mathbf{v}_d(x)$ be the vector of monomials $(x^{\alpha})$ of degree $d$, i.e., such that $\alpha_1 + \cdots + \alpha_n = d$. (And so $\mathbf{v}_1(x) = x$.)

If $g \in \mathbb{R}[x]_{2d}$ is homogeneous and SOS then

$$g(x) = \frac{1}{2} \mathbf{v}_d(x)^T \Sigma \mathbf{v}_d(x),$$

for some real symmetric positive semidefinite matrix $\Sigma \succeq 0$.

And if $d = 1$ one has the Gaussian property

$$\int_{\mathbb{R}^n} \exp(-g) \, dx = \frac{(2\pi)^{n/2}}{\sqrt{\det \Sigma}},$$
$$\int_{\mathbb{R}^n} \mathbf{v}_d(x) \mathbf{v}_d(x)^T \exp(-g) \, dx \quad \int_{\mathbb{R}^n} \exp(-g) \, dx = \Sigma^{-1}.$$
Let \( v_d(x) \) be the vector of monomials \((x^\alpha)\) of degree \(d\), i.e., such that \(\alpha_1 + \cdots + \alpha_n = d\). (And so \(v_1(x) = x\).)

If \( g \in \mathbb{R}[x]_{2d} \) is homogeneous and SOS then

\[
g(x) = \frac{1}{2} v_d(x)^T \Sigma v_d(x),
\]

for some real symmetric positive semidefinite matrix \(\Sigma \succeq 0\).

And if \(d = 1\) one has the Gaussian property

\[
\int_{\mathbb{R}^n} \exp(-g) \, dx = \frac{(2\pi)^{n/2}}{\sqrt{\det \Sigma}},
\]

\[
\frac{\int_{\mathbb{R}^n} v_d(x) v_d(x)^T \exp(-g) \, dx}{\int_{\mathbb{R}^n} \exp(-g) \, dx} = \Sigma^{-1}.
\]
In other words, if $\mu$ is the Gaussian measure

$$
\mu(B) := \frac{\int_B \exp\left(-\frac{1}{2} x^T \Sigma x\right) \, dx}{\int_{\mathbb{R}^n} \exp\left(-\frac{1}{2} x^T \Sigma x\right) \, dx}, \quad \forall B,
$$

then its (covariance) matrix of moments of order 2 satisfies:

$$
M_1(\Sigma) := \int_{\mathbb{R}^n} x x^T \, d\mu(x) = \Sigma^{-1},
$$

and the function

$$
\theta_1(\Sigma) := (\det \Sigma)^{1/2} \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2} v_1(x)^T \Sigma v_1(x)\right) \, dx.
$$

is constant!

... not true anymore for $d > 1!$
However, let $\ell(d) = \binom{n+d-1}{d}$, and $S_{++}^{\ell(d)}$ be the cone of real positive definite $\ell(d) \times \ell(d)$ matrices. Let $k := n/(2d\ell(d))$.

With $\Sigma \in S_{++}^{\ell(d)}$, define the probability measure $\mu$

$$
\mu(B) := \frac{\int_B \exp \left(-k\mathbf{v}_d(x)^T \Sigma \mathbf{v}_d(x) \right) \, dx}{\int_{\mathbb{R}^n} \exp \left(-k\mathbf{v}_d(x)^T \Sigma \mathbf{v}_d(x) \right) \, dx}, \quad \forall B,
$$

with matrix of moments of order $2d$ given by:

$$
M_d(\Sigma) := \int_{\mathbb{R}^n} \mathbf{v}_d(x) \mathbf{v}_d(x)^T \, d\mu(x).
$$
Define $\theta_d : S_{++}^{(d)} \rightarrow \mathbb{R}$ to be the function

$$\Sigma \mapsto \theta_d(\Sigma) := (\det \Sigma)^k \int_{\mathbb{R}^n} \exp \left( -k \mathbf{v}_d(x)^T \Sigma \mathbf{v}_d(x) \right) \, dx.$$ 

**Theorem**

$$M_d(\Sigma) = \Sigma^{-1} \iff \nabla \theta_d(\Sigma) = 0$$

Hence critical points $\Sigma^*$ of $\theta_d$ have the Gaussian property

$$\frac{\int \mathbf{v}_d(x) \mathbf{v}_d(x)^T \exp \left( -k \mathbf{v}_d(x)^T \Sigma^* \mathbf{v}_d(x) \right) \, dx}{\int \exp \left( -k \mathbf{v}_d(x)^T \Sigma^* \mathbf{v}_d(x) \right) \, dx} = (\Sigma^*)^{-1}$$

☆ If $d = 1$ then $\theta_d(\cdot)$ is constant and so $\nabla \theta_d(\cdot) = 0$.
☆ If $d > 1$ then $\theta_d(\cdot)$ is constant in each ray $\lambda \Sigma$, $\lambda > 0$. 

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\[ \nabla \theta_d(\Sigma) = k \frac{\Sigma^A}{\det \Sigma} \theta_d(\Sigma) \]

\[ -k(\det \Sigma)^k \int_{\mathbb{R}^n} v_d(x)v_d(x)^T \exp \left( -kv_d(x)^T \Sigma v_d(x) \right) \, dx \]

\[ = k\theta_d(\Sigma) \left[ \Sigma^{-1} - M_d(\Sigma) \right] \]

and so

\[ M_d(\Sigma) = \Sigma^{-1} \implies \nabla \theta_d(\Sigma) = 0. \]
If $K \subset \mathbb{R}^n$ is compact then computing the ellipsoid $\xi$ of minimum volume containing $K$ is a classical problem whose optimal solution is called the Löwner-John ellipsoid. So consider the following problem:

Find an homogeneous polynomial $g \in \mathbb{R}[x]_{2d}$ such that its sublevel set $G := \{x : g(x) \leq 1\}$ contains $K$ and has minimum volume among all such levels sets with this inclusion property.
Let $P[x]_{2d}$ be the convex cone of homogeneous polynomials of degree $2d$ whose sub-level set $G = \{ x : g(x) \leq 1 \}$ has finite Lebesgue volume and with $K \subset \mathbb{R}^n$, let $C_{2d}(K)$ be the convex cone of polynomials nonnegative on $K$.

**Lemma**

Let $K \subset \mathbb{R}^n$ be compact. The minimum volume of a sublevel set $G = \{ x : g(x) \leq 1 \}$, $g \in P[x]_{2d}$, that contains $K \subset \mathbb{R}^n$ is $\rho / \Gamma(1 + n/2d)$ where:

$$P : \quad \rho = \inf_{g \in P[x]_{2d}} \left\{ \int_{\mathbb{R}^n} \exp(-g) \, dx : 1 - g \in C_{2d}(K) \right\}.$$

a finite-dimensional convex optimization problem!
Let $P[x]_{2d}$ be the convex cone of homogeneous polynomials of degree $2d$ whose sub-level set $G = \{ x : g(x) \leq 1 \}$ has finite Lebesgue volume and with $K \subset \mathbb{R}^n$, let $C_{2d}(K)$ be the convex cone of polynomials nonnegative on $K$.

Lemma

Let $K \subset \mathbb{R}^n$ be compact. The minimum volume of a sublevel set $G = \{ x : g(x) \leq 1 \}$, $g \in P[x]_{2d}$, that contains $K \subset \mathbb{R}^n$ is $\rho/\Gamma(1 + n/2d)$ where:

$$\mathcal{P} : \rho = \inf_{g \in P[x]_{2d}} \left\{ \int_{\mathbb{R}^n} \exp(-g) \, dx : 1 - g \in C_{2d}(K) \right\}.$$ 

a finite-dimensional convex optimization problem!
We have seen that:
\[
\text{vol} \left( \{ x : g(x) \leq 1 \} \right) = \frac{1}{\Gamma(1 + n/2d)} \int_{\mathbb{R}^n} \exp(-g) \, dx.
\]
Moreover, the sub-level set \( \{ x : g(x) \leq 1 \} \) contains \( K \) if and only if \( 1 - g \in C_{2d}(K) \), and so \( \rho / \Gamma(1 + n/2d) \) is the minimum value of all volumes of sub-levels sets \( \{ x : g(x) \leq 1 \} \), \( g \in P[x]_{2d} \), that contain \( K \).

Now since \( g \mapsto \int_{\mathbb{R}^n} \exp(-g) \, dx \) is strictly convex and \( C_{2d}(K) \) is a convex cone, problem \( P \) is a finite-dimensional convex optimization problem. \( \square \)
Theorem

(a) $\mathcal{P}$ has a unique optimal solution $g^* \in \mathbb{P}[x]_{2d}$ and there exists a Borel measure $\mu^*$ supported on $K$ such that:

\[
\begin{aligned}
\int_{\mathbb{R}^n} x^\alpha \exp(-g^*) \, dx &= \int_K x^\alpha \, d\mu^*, \quad \forall |\alpha| = 2d \\
\int_K (1 - g^*) \, d\mu^* &= 0
\end{aligned}
\]

In particular, $\mu^*$ is supported on the real variety $V := \{x \in K : g^*(x) = 1\}$ and in fact, $\mu^*$ can be substituted with another measure $\nu^*$ supported on at most \(\binom{n+2d-1}{2d}\) points of $V$.

(b) Conversely, if $g^* \in \mathbb{P}[x]_{2d}$ and $\mu^*$ satisfy (*) then $g^*$ is an optimal solution of $\mathcal{P}$.
Theorem

(a) $P$ has a unique optimal solution $g^* \in P[x]_{2d}$ and there exists a Borel measure $\mu^*$ supported on $K$ such that:

\[
\begin{cases}
\int_{\mathbb{R}^n} x^\alpha \exp(-g^*) \, dx = \int_K x^\alpha \, d\mu^*, & \forall |\alpha| = 2d \\
\int_K (1 - g^*) \, d\mu^* = 0
\end{cases}
\]

In particular, $\mu^*$ is supported on the real variety $V := \{ x \in K : g^*(x) = 1 \}$ and in fact, $\mu^*$ can be substituted with another measure $\nu^*$ supported on at most $\binom{n+2d-1}{2d}$ points of $V$.

(b) Conversely, if $g^* \in P[x]_{2d}$ and $\mu^*$ satisfy (*) then $g^*$ is an optimal solution of $P$. 

Example

Let \( K \subset \mathbb{R}^2 \) be the box \([-1, 1]^2\).

The set \( G_4 := \{ x : g(x) \leq 1 \} \) with \( g \) homogeneous of degree 4 which contains \( K \) and has minimum volume is

\[
x \mapsto g_4(x) := x_1^4 + y_1^4 - x_1^2 x_2^2,
\]

with \( \text{vol}(G_4) \approx 4.39 \) much better than
- \( \pi R^2 \approx 2\pi \approx 6.28 \) for the Löwner-John ellipsoid of minimum volume, and
- the (convex) TV screen \( G := \{ x : (x_1^4 + x_2^4)/2 \leq 1 \} \) with volume \( > 5 \).
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Recovery of algebraic-exponential data from moments
Let $\mathbf{K} \subset \mathbb{R}^2$ be the box $[-1, 1]^2$.

The set $G_6 := \{ \mathbf{x} : g(\mathbf{x}) \leq 1 \}$ with $g$ homogeneous of degree 6 which contains $\mathbf{K}$ and has minimum volume is

$$\mathbf{x} \mapsto g_6(\mathbf{x}) := x_1^6 + y_1^6 - (x_1^4 x_2^2 + x_1^2 x_2^4)/2,$$

with $\text{vol}(G_6) \approx 4.19$ much better than

- $\pi R^2 = 2\pi \approx 6.28$ for the Löwner-John ellipsoid of minimum volume, and
- better than the set $G_4$ with volume 4.39.
Recovery of algebraic-exponential data from moments
Write \( g(x) = \sum_{\beta} g_\beta x^\beta \) and let \( G := \{ x : g(x) \leq 1 \} \).

**Lemma**

If \( g \) is nonnegative and \( d \)-homogeneous with \( \text{vol}(G) < \infty \) then:

\[
\begin{align*}
\int_{G} x^\alpha g(x) \, dx &= \frac{n + |\alpha|}{n + d + |\alpha|} \int_{G} x^\alpha \, dx, \\
\sum_{\beta} g_\beta y_{\alpha + \beta} &= n + d + |\alpha|
\end{align*}
\]

and so we see that the moments \( (y_\alpha) \) satisfy linear relationships explicit in terms of the coefficients of the polynomial \( g \) that describes the boundary of \( G \).
So let us write \( g \in \mathbb{R}^{s(d)} \) the unknown vector of coefficients of the unknown polynomial \( g \).
Let \( M_d(y) \) be the moment matrix of order \( d \) whose rows and columns are indexed in the canonical basis of monomials \( (x^\alpha) \), \( \alpha \in \mathbb{N}_d^n \), and with entries

\[
M_d(y)(\alpha, \beta) = y_{\alpha+\beta}, \quad \alpha, \beta \in \mathbb{N}_d^n.
\]

and let \( y^d \) be the vector \( (y_\alpha) \), \( \alpha \in \mathbb{N}_d^n \).

Previous Lemma states that

\[
M_d(y)g = y^d,
\]

or, equivalently,

\[
g = M_d(y)^{-1} y^d,
\]

because the moment matrix \( M_d(y) \) is nonsingular whenever \( G \) has nonempty interior.
In other words ...

one may recover \( g \) EXACTLY from knowledge of moments \((y_\alpha)\) of order \( d \) and \( 2d! \)
Non homogeneous polynomials

Given a polynomial \( g \in \mathbb{R}[x]_d \) write \( g(x) = \sum_{k=0}^{d} g_k(x) \), where each \( g_k \) is homogeneous of degree \( k \).

Lemma

Let \( g \in \mathbb{R}[x]_d \) be such that its level set \( G := \{ x : g(x) \leq 1 \} \) is bounded. Then for every \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \):

\[
\int_G x^\alpha (1 - g(x)) \, dx = \sum_{k=1}^{d} \frac{k}{n + |\alpha|} \int_G x^\alpha g_k(x) \, dx
\]

Proof: Use Stokes’ formula

\[
\int_G \text{Div}(X) f(x) \, dx + \int_G \langle X, \nabla f(x) \rangle \, dx = \int_{\partial G} \langle X, \vec{n}_x \rangle f \, d\sigma,
\]

with vector field \( X = x \) and \( f(x) = x^\alpha (1 - g(x)) \).
Then observe that $\text{Div}(X) = n$ and:

$$\langle X, \nabla f(x) \rangle = |\alpha| f - x^\alpha \sum_{k=1}^{d} k g_k(x).$$

In the general case, when $\partial G$ may have singular points, or lower dimensional components, we can invoke Sard’s theorem, for the (smooth) sublevel sets

$$G_\gamma = \{ x : g(x) < \gamma \}$$

and pass to the limit $\gamma \to 1$, $\gamma < 1$. □
Let $G \subset \mathbb{R}^n$ be open with $G = \text{int} \overline{G}$ and with real algebraic boundary $\partial G$. A polynomial of degree $d$ vanishes on $\partial G$.

Define a renormalised moment-type matrix $M^d_k(y)$ as follows:

- $s(d) \left(= \binom{n+d}{n}\right)$ columns indexed by $\beta \in \mathbb{N}_d^n$,
- countably many rows indexed by $\alpha \in \mathbb{N}_k^n$,
and with entries:

$$M^d_k(y)(\alpha, \beta) := \frac{n + |\alpha| + |\beta|}{n + |\alpha|} y^{\alpha+\beta}, \quad \alpha \in \mathbb{N}_k^n, \beta \in \mathbb{N}_d^n.$$
Theorem

Let $G \subset \mathbb{R}^n$ be a bounded open set with real algebraic boundary. Assume that $G = \text{int} \overline{G}$ and a polynomial of degree $d$ vanishes on $\partial G$ and not at 0. Then the linear system

$$M_{2d}^d(y) \begin{bmatrix} -1 \\ g \end{bmatrix} = 0,$$

admits a unique solution $g \in \mathbb{R}^{s(d) - 1}$, and the polynomial $g$ with coefficients $(0, g)$ satisfies

$$(x \in \partial G) \Rightarrow (g(x) = 1).$$
The identity (obtained from Stokes’ theorem)

\[
\int_{\mathbf{G}} \mathbf{x}^\alpha (1 - g(\mathbf{x})) \, d\mathbf{x} = \sum_{k=1}^{d} \frac{k}{n + |\alpha|} \int_{\mathbf{G}} \mathbf{x}^\alpha g_k(\mathbf{x}) \, d\mathbf{x}
\]

for all \(\alpha \in \mathbb{N}_k^n\)

in fact reads:

\[
M^d_k(y) \begin{bmatrix} -1 \\ \mathbf{g} \end{bmatrix} = 0,
\]

Conversely, if \(g\) solves

\[
M^d_{2d}(y) \begin{bmatrix} -1 \\ \mathbf{g} \end{bmatrix} = 0,
\]

then

\[
\int_{\partial \mathbf{G}} \langle \mathbf{x}, \vec{n}_x \rangle (1 - g(\mathbf{x})) \mathbf{x}^\alpha \, d\sigma = 0, \quad \forall \alpha \in \mathbb{N}_{2d}^n.
\]
The identity (obtained from Stokes’ theorem)

\[
\int_{\mathcal{G}} x^\alpha (1 - g(x)) \, dx = \sum_{k=1}^{d} \frac{k}{n + |\alpha|} \int_{\mathcal{G}} x^\alpha g_k(x) \, dx
\]

for all \( \alpha \in \mathbb{N}_k^n \)

in fact reads:

\[
M^{d}_k(y) \begin{bmatrix} -1 \\ g \end{bmatrix} = 0,
\]

Conversely, if \( g \) solves

\[
M^{d}_{2d}(y) \begin{bmatrix} -1 \\ g \end{bmatrix} = 0,
\]

then

\[
\int_{\partial \mathcal{G}} \langle x, \vec{n}_x \rangle (1 - g(x)) x^\alpha \, d\sigma = 0, \quad \forall \alpha \in \mathbb{N}^{n}_{2d}.
\]
As $\partial G$ is algebraic, one may write

$$\vec{n}_x = \frac{\nabla h(x)}{\|\nabla h(x)\|},$$

for some polynomial $h$. Therefore

$$0 = \int_{\partial G} \langle x, \vec{n}_x \rangle (1 - g(x)) x^\alpha d\sigma \quad \forall \alpha \in \mathbb{N}_{2d}$$

$$= \int_{\partial G} \langle x, \nabla h(x) \rangle (1 - g(x)) x^\alpha \frac{1}{\|\nabla h\|} d\sigma' \quad \forall \alpha \in \mathbb{N}_{2d}$$

$$\Rightarrow \int_{\partial G} \langle x, \nabla h(x) \rangle^2 (1 - g(x))^2 d\sigma' = 0 \quad \square$$
As $\partial G$ is algebraic, one may write

$$\vec{n}_x = \frac{\nabla h(x)}{\|\nabla h(x)\|},$$

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$$0 = \int_{\partial G} \langle x, \vec{n}_x \rangle (1 - g(x)) x^\alpha d\sigma \quad \forall \alpha \in \mathbb{N}^n_{2d}$$

$$= \int_{\partial G} \left\langle x, \nabla h(x) \right\rangle (1 - g(x)) x^\alpha \frac{1}{\|\nabla h\|} d\sigma' \quad \forall \alpha \in \mathbb{N}^n_{2d}$$

$$\Rightarrow \int_{\partial G} \left\langle x, \nabla h(x) \right\rangle^2 (1 - g(x))^2 d\sigma' = 0 \quad \square$$
For sake of rigor the boundary $\partial G$ can be written

\[ \partial G = Z_0 \cup Z_1, \]

with $Z_0$ being a finite union of smooth $n-1$-submanifolds of $\mathbb{R}^n$ leaving $G$ on one side, $Z_1$ is a union of the lower dimensional strata, and $\sigma(Z_1) = 0$. 
Convexity

Theorem

Let $G \subset \mathbb{R}^n$ be a bounded convex open set with real algebraic boundary. Assume that $G = \text{int} \ G$, $0 \in G$, and a polynomial of degree $d$ vanishes on $\partial G$ and not at $0$. Then the linear system

$$M_d(y) \begin{bmatrix} -1 \\ g \end{bmatrix} = 0,$$

admits a unique solution $g \in \mathbb{R}^{s(d)-1}$, and the polynomial $g$ with coefficients $(0, g)$ satisfies

$$(x \in \partial G) \Rightarrow (g(x) = 1).$$
As in the previous proof, if

$$M_d^d(y) \begin{bmatrix} -1 \\ g \end{bmatrix} = 0,$$

then

$$\int_{\partial G} \langle x, \vec{n}_x \rangle (1 - g(x))^2 \, d\sigma = 0.$$ 

But one now uses that if $0 \in G$ then $\langle x, \vec{n}_x \rangle \geq 0$. 
A consequence in Probability

Consider the Probability measure $\mu$

uniformly supported on a set $G$ of the form $\{x : g(x) \leq 1\}$, for some polynomial $g \in \mathbb{R}[x]_d$.

Then:

- ALL moments $y_\alpha := \int_G x^{\alpha} d\mu$, $\alpha \in \mathbb{N}^n$, are determined from those up to order $3d$ (and $2d$ if $G$ is convex)!

- A similar result holds true if now $\mu$ has a density $\exp(h(x))$ on $G$ (for some $h \in \mathbb{R}[x]$).

→ is an extension to such measures of a well-known result for exponential families.
A consequence in Probability

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Recovery of algebraic-exponential data from moments
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- A similar result holds true if now $\mu$ has a density $\exp(h(x))$ on $G$ (for some $h \in \mathbb{R}[x]$).

→ is an extension to such measures of a well-known result for exponential families
Conclusion

- Compact sub-level sets \( G := \{ x : g(x) \leq y \} \) of homogeneous polynomials exhibit surprising properties. E.g.:
  - convexity of \( \text{volume}(G) \) with respect to the coefficients of \( g \)
  - Integrating a PHF \( h \) on \( G \) reduce to evaluating the non Gaussian integral \( \int h \exp(-g)dx \)
  - A variational property yields a Gaussian-like property
  - exact recovery of \( G \) from finitely moments.
    (Also works for quasi-homogeneous polynomials with bounded sublevel sets!)
  - exact recovery for sets with algebraic boundary of known degree
Practical and important issues

- **COMPUTATION!**: Efficient evaluation of $\int_{\mathbb{R}^n} \exp(-g) \, dx$, or equivalently, evaluation of $\text{vol}\left(\{x : g(x) \leq 1\}\right)$!

- The property

  $$\int_G x^\alpha g(x) \, dx = \frac{n + |\alpha|}{n + d + |\alpha|} \int_G x^\alpha \, dx, \quad \forall \alpha,$$

  helps a lot to improve efficiency of the method in Henrion, Lasserre and Savorgnan (SIAM Review)
THANK YOU!